Matrix

Array – Collection of numbers arranged in rows and columns.

→ A Matrix is a rectangular array of no's closed in addition, substraction and multiplication, devision.

$$A = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Some Particular Matrix

(i) Square Matrix: Matrix with no of Rows = no of Columns.

$$A_{m\times n} \Rightarrow m = n$$

(ii) Diagonal Matrix: $A = [a_{ij}]$ such that $a_{ij} = 0 \ \forall\ i \neq j$

ie $A = \text{diag}[d_1, d_2, \ldots d_n]$

Zero Matrix is a Diagonal Matrix

Note - here condition is on non-diagonal elements.

(III) Scalar Matrix: A special kind of diagonal matrix.

such that $A = [a_{ij}]$, $a_{ij} = 0 \ \forall\ i \neq j$

& $a_{ij} = k \ \forall\ i = j$

here all the diagonal elements are same entry.

(iv) Triangular Matrix: all the entries above or below the diagonal of a square matrix are zero, then it is a triangular matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Upper Triangular Matrix | Lower Triangular Matrix

Equality of Matrices

Matrix A and B are equal ~~of~~ if they have same orders and respective elements are equal.

Transpose of a Matrix

$A = [a_{ij}]_{m\times n}$ then $A^T = [a_{ji}]_{n\times m}$

(i) $(A^T)^T = A$ (ii) $(A+B)' = A' + B'$

(iii) $(AB)' = B'A'$

Symmetric and Skew-symmetric Matrix

Symmetric if $A^T = A$ ie $a_{ij} = a_{ji}$

→ must be a square matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 5 \end{bmatrix}$$

(*) If A and B are symmetric matrix.

(i) $\lambda A + \mu B$ is also symmetric ; $\lambda, \mu \in$ scalars

(ii) A^n is also symmetric $\forall$ +ve integral powers

(iii) $A + A'$ or AA' are also symmetric $\forall A$, square matrix

(iv) AB

(v) $AB + BA$

Skew Symmetric if $A' = -A$ ie $a_{ij} = -a_{ji}$

and $a_{ii} = 0$

diagonal entries are 0

* $\boxed{\text{Zero Matrix is both sym. and skew-sym.}}$

if A and B are skew-sym matrices

(i) AB is skew symmetric if $AB = -BA$

(ii) $AB - BA$ is skew symmetric

(iii) λA is skew sym. λ any scalar

(iv) $A - A'$ is skew-sym $\forall A$ (square matrix)

*(v) $A = \underbrace{\frac{1}{2}(A + A')}_{\text{sym.}} + \underbrace{\frac{1}{2}(A - A')}_{\text{skew-sym}}$

*(vi) $A \longrightarrow A$ skew sym.

$A^K \longrightarrow$ skew sym $\forall K =$ +ve odd integer

$A^K \longrightarrow$ sym $\forall K =$ +ve even integer.

Conjugate Matrix ($\bar{A}$)

Matrix obtained by replacing each element of a matrix A by its complex conjugate.

if $\bar{A} = A \longrightarrow$ A is said to be Real Matrix

Hermition and Skew-Hermition Matrix

$\boxed{A^{\theta} = A}$; $A^{\theta} = (\bar{A})'$ or $(\overline{A'})$

eg. $A = \begin{bmatrix} 1 & 2+i \\ 2-i & 2 \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} 1 & 2-i \\ 2+i & 2 \end{bmatrix}$

$\Rightarrow (\bar{A})' = \begin{bmatrix} 1 & 2+i \\ 2-i & 2 \end{bmatrix}$ $\therefore A^{\theta} = A$.

so, $A = [a_{ij}]$ s.t. $\bar{a}_{ij} = a_{ji}$ & $\bar{a}_{ii} = a_{ii}$

now, as, $\bar{a}_{ii} = a_{ii} \longrightarrow$ Real No

$\boxed{A^{\theta} = -A} \Leftarrow$ Skew-Hermition

if $A = [a_{ij}]$ then $\bar{a}_{ij} = -a_{ji}$
& $\bar{a}_{ii} = -a_{ii}$] — skew-hermition

imaginary entry $(i, 2i, -i)$ or '0'.

Note :-

(i) Every square matrix can be uniquely represented as the sum of hermition and skew-hermition matrix

(ii) Every square matrix can be uniquely represented as $P+Q$ where P and Q are hermition.

(iii) if $A \longrightarrow$ Hermition then iA is a Skew-hermition

$A \longrightarrow$ Skew-hermition then iA is Hermition.

Idempotent Matrix :

A square matrix A such that $A^2 = A$ and

if $A' = A$ and $A^2 = A$

then A is symmetric idempotent

* Every non-singular matrix (idempotent) is an identity matrix.

Non-Singular $\Rightarrow \det(A) \neq 0$

Singular $\Rightarrow \det(A) = 0$

Nilpotent Matrix

$A^n = 0$ for some $n \in \mathbb{N}$ then A is nilpotent

Index of Matrix, if $A^m = 0$ and $A^k \neq 0$ $\forall k \leq m$.

then m is index.

* Nilpotent Matrix is Singular

Nilpotent $\Rightarrow$ Singular

Non-Singular $\Rightarrow$ Non-nilpotent

Involutory Matrix

A is involutory matrix

$A^2 = I$ and its determinant is ± 1

if A and B are involutory,

(i) $A+B$ is not involutory

(ii) AB is involutory if $AB = BA$

(iii) A^n is involutory

* Orthogonal Matrix

A square matrix A is orthogonal if $A'A = AA' = I$

$\det(A) = \pm 1$

if $|A| = 1$, A is a Proper Matrix

Note:- A, B are Orthogonal Matrix

(i) $A \cdot B$ is Orthogonal

(ii) A^n is "

(iii) A' " "

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☑ Unitory Matrix :

A square matrix A is said to be Unitory if $A^\theta A = A A^\theta = I$

$\rightarrow \det(A) = \text{mod } 1$

if A is a real matrix then, $A^\theta = A'$

ie. if A is Unitory Matrix then, $AA' = A'A = I$

(*) Strict Upper Triangular Matrix / Strict Lower Triangular Matrix are always Nilpotent with index n.

☑ Traces of Matrix

sum of elements of principal diagonal elements.

$A = [a_{ij}]_n$

$tr(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$

(i) $tr(ABC) = tr(BCA) = tr(CAB)$

(ii) $tr(AB)^K = tr(BA)^K \neq tr(A^K B^K)$

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Field

for F to be Field, every non-zero element 'K' has a multiplicative inverse, ie

$$K \cdot \frac{1}{K} = 1$$

- Natural No ⟶ Field ✗ ⟵ Integer
- Rational No, Real No ⟶ Field ✓ ⟵ Complex No

Vector Space

Let F be a field, a non-empty set B together with two binary operation – vector addition & scalar multiplication is called a vector space over field F if,

the following holds –

1) V is an additive abelian grp

(i) closure : $u, v \in V$ then $u+v \in V$.

(ii) associative :

$$u+(v+w) = (u+v)+w \quad \forall \; u, v, w \in V$$

(iii) identity :

$$u+\bar{0} = \bar{0}+u = u$$

(iv) inverse :

$$u+(-u) = (-u)+u = 0$$

(v) cummutative : $u+v = v+u$

2) V is closed under scalar multiplication

$$au \in V \;\; \forall \; u \in V \;\&\; a \in F.$$

3) $a(u+v) = au + av$

4) $(a+b).u = au + bu$

5) $(ab)u = a(bu)$

6) $1.u = u$ where 1 is multiplicative identity

Denoting Vector Space

$C(R)$ — R ↓ field

$C \longrightarrow$ set of complex no.

$R \longrightarrow$ set of Real no.

* Every field is a vector space over itself

ie $F(F) \longrightarrow$ Vector Space

* Every field is a vector space over its subfield

$$N \subseteq Z \subseteq Q \subseteq R \subseteq C$$

* $F^n(F) \longrightarrow$ Vector Space

Subspace

It is a subset of $V(F)$ such that it is itself a Vector Space over F.

A nonempty subset 'W' of $V(F)$ is called subspace of V if W is closed under :

i) addition $\rightarrow a \in W, b \in W \Rightarrow a+b \in W$

ii) multiplication $\rightarrow \lambda \in F, a \in W \Rightarrow \lambda a \in W$.

Note: Vector Space V and zero space $\langle 0 \rangle$ are always subspaces of 'V' and are called improper subspaces.

- Necessary and Sufficient condn for W to be subspace of V :

i) $0 \in W$ ii) $u+v \in W \ \forall \ u, v \in W$

iii) $au \in W$; $a \in F$ & $u \in W$

Remarks: (i) The intersection of two subspaces is a subspace

(ii) Union of two subspace is a Subspace iff one is contented in other.

* Consider a set $\langle a_1, a_2, \ldots a_n \rangle$ of vectors from vector space 'V' then the collection of all possible linear combinations of $\langle a_1 \ldots a_n \rangle$ is a vector subspace of V.

Linear Sum.	Direct Sum
(i) W_1 and $W_2 \longrightarrow$ subspaces of V. Then, Linear Sum $= W_1 + W_2$ and $W_1 + W_2 = w_1 + w_2$; $w_1 \in W_1$, $w_2 \in W_2$	(i) for vector space 'V' to be direct sum of its subspaces W_1 and W_2. a) $V = W_1 + W_2$ b) $W_1 \cap W_2 = \langle 0 \rangle$
(ii) Linear Sum is a Subspace	(ii) Every direct sum is a Linear Sum; " " " " " Subspace.

☑ Linear Combination

$S \subseteq V(F)$ ie S is subset of $V(F)$

$S = \langle v_1, v_2, \dots v_n \rangle$

let, $v \in V(F)$

if $v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n \quad \forall\ a_i, v_i \in F$

then v is linear combination of $v_1, v_2 \dots v_n$.

☑ Spanning Set (Generating Set)

A set $\langle a_1, a_2 \dots a_n \rangle$ of vectors from V is said to be ~~the~~ Span if, every vector in V can be written as a Linear Combination of $a_1, a_2 \dots a_n$.

→ Collection of all possible L.C. of $\langle a_1, a_2 \dots a_n \rangle$ is

$$S = \left\langle \sum_{i=1}^{k} l_i a_j \ , l_i \in R \right\rangle \text{ is a Vector Space}$$

eg. $V = R^3(R)$

spanning set, $S = \langle a_1, a_2, a_3 \rangle$

$a_1 = (1,0,0)$; $a_2 = (0,1,0)$; $a_3 = (0,0,1)$

☑ Linearly Dependent Vectors :

$S \subseteq V(F)$ namely,

$S = \langle x_1, x_2, \dots x_n \rangle$ is L.D. if—

$\exists\ a_1, a_2, \dots a_n$ not all zero

such that— $a_1 x_1 + a_2 x_2 + \dots + a_n x_n = 0$

☑ Linearly Independent

$S \subset V(F)$; $S = \langle x_1, x_2, \dots x_n \rangle$ is L.I. if—

$\exists\ a_1, a_2, \dots a_n$ such that—

$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = 0$

$\Rightarrow a_1 = a_2 = \dots = a_n = 0$

☑ Method to Check L.D or L.I

$S = \langle v_1, v_2, \dots v_n \rangle$

→ form a matrix

$A = [v_1 \ v_2 \ .. \ v_n]$

if $\det(A) = 0 \longrightarrow$ Linearly Dependent

$\det(A) \neq 0 \longrightarrow$ Linearly Independent

☑ Basis of a Vector Space

$V(F) \longrightarrow$ V.S.

a set of vectors $\langle v_1, v_2 .. v_n \rangle \in V$ is called Basis of V if

i) $v_1, v_2 \dots v_n \longrightarrow$ Linearly indept

ii) $v_1, v_2, \dots, v_n$ spans V.

(*) Basis are not unique

(*) $S = \langle v_1, v_2 \dots v_n \rangle$ spans V then $\exists$ a subset of 'S' which is a Basis of 'V'.

☑ Dimension

The number of elements in ~~every~~ any basis of a vector space V is called Dimension of 'V' and denoted by ($\dim V$)

(Extension) → If V is a finitely generated V.S. then, any set of Linearly Indept vectors can be extended to a basis.

eg. $V = R^3(R)$

$S = \langle (0,0,1), (0,1,0) \rangle$

extend → $\langle (1,0,0), (0,0,1), (0,1,0) \rangle$

☑ Dimension of Subspace

first determine the basis of a subspace

Generating set is given

$W \longrightarrow$ Subspace of V

and $S = \langle v_1, v_2 .. v_n \rangle$ G.S. of W

i) write given vectors as row of a matrix A

ii) Reduce 'A' into row-echlon form

iii) Non-zero rows of row-echlon form gives the basis of subspace

Let W_1 and W_2 are two subspaces of 'V'. V is finite dimentional then

$$\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

Row-Echlon Form

→ all zero rows if any are at the bottom

→ each first ~~row~~ non-zero entry of any row is 1

→ all entries in a column below a leading 1 are zero.

Dimension of Vector Space

V → Finitely Generated Vector Space

i) select the most general form of a vector

ii) express it in terms of elements of the field

iii) note the no of unknowns which we can freely choose

eg. $V \longrightarrow C^n(R)$

$V = \langle z_1, z_2, \dots z_n \rangle$

$\Rightarrow V = \langle a_1 + ib_1,\ a_2 + ib_2, \dots a_n + ib_n \rangle$

$a_1, a_2, \dots a_n \quad \exists \to n$

$b_1, b_2 \dots b_n \quad \exists \to n$

$2n \longrightarrow \dim(V)$.

Rank of a Matrix

$A_{m \times n} \longrightarrow$ Matrix

if r is the Rank of A, if

i) the determinant of any (atleast) one minor of order 'r' is non-zero.

ii) the determinant of every minor of order higher than r is zero

→ sub matrix

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

Rank using Row-Echlon form

No of nonzero rows in the row echlon forms of a Matrix gives its Rank.

<u>Practical Use</u>: to check if given vectors are Linearly Dept or indepl —

St1 → construct Matrix A by writting given vectors in column

St2 → convert A to its Row-Echlon form

St3 → (i) if $\rho(A)$ = no of columns in A —

↳ vectors are LI.

(ii) if $\rho(A) <$ no of columns in A —

↳ vectors are L.D.

Properties of Rank

(i) only rank of Null Matrix is Zero

(ii) $\rho(I_n) = n$; I_n = unit matrix of order n.

(iii) $\rho(A_{mn}) \leq \min(m,n)$

(iv) $\rho(A_{n \times n}) = n$ if $|A| \neq 0$

$< n$ if $|A| = 0$

(v) if $\rho(A) = m$, $\rho(B) = n$ then $\rho(AB) \leq \min(m,n)$

(vi) if A and B are square matrix of order n

$$\rho(AB) \geq \rho(A) + \rho(B) - n$$

*(vii) if A^{-1} exists, and B is a matrix of any order then,

$\rho(AB)$ doesn't depend on A

ie. $\rho(AB) = \rho(B)$.

(viii) $\rho(A') = \rho(A)$

System of Linear Equation

Linear Equation — A Linear Eq. is n variables $x_1, x_2 \ldots x_n$ is any equation in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

Where $a_1, a_1 \ldots a_n, b \rightarrow$ Real / Complex

→

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System of Linear Equations.

m-equations in n-variables

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{\text{Coefficient Matrix}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

$$\boxed{AX = B}$$

$C = [A : B] \longrightarrow$ Augmented Matrix

System of Eq.

- Homogeneous.
  $AX = B$
  if $b_1 = b_2 = \cdots = b_m = 0$
  ↓
  homogeneous system

- Non-Homogeneous
  $AX = B$
  if atleast one b_i is not equal to zero.

System → Consistent — (if solution exists)
System → Non Consistent — (if no solution exists)

Consistent → unique solution
Consistent → infinitely many solution

Non-Homogeneous System : may have unique solution, infinitely many solution or no solution

Homogeneous System is never inconsistent.

Getting Solutions:

- **Homogeneous System** $\rightarrow$

  (I) if $\rho(A) = n$ ie no of unknowns,

  then system has unique solution (trivial Solution)

  (ii) if $\rho(A) < n$, then system has ∞ no of non-trivial solution.

- **Non-Homogeneous System** $\rightarrow$

  $$AX = B \Rightarrow C = [A:B]$$

  (i) if $\rho(C) \neq \rho(A) \longrightarrow$ inconsistent ie no solution

  (ii) if $\rho(C) = \rho(A) =$ no of unknowns then system has unique solution.

  (iii) if $\rho(C) = \rho(A) =$ less than no of unknowns; system has infinite no of solutions.

  (iv) $X = 0 \longrightarrow$ is never solution of non-homo system.

Linear Transformation.

V and W $\exists \to$ two vectorspaces over field F (may have different dimentions)

then a mapping, $T: V \longrightarrow W$ is called Linear Transformation if,

(i) $T(x+y) = T(x) + T(y) \quad \forall\ x, y \in V$

necessary condition (ii) $T(ax) = aT(x) \quad \forall\ a \in F\ \&\ x \in V$

* if $T(V) \longrightarrow W$ is a Linear Transformation then, 'T' takes the zero vector of V into zero vector of W

Zero Linear Transformation

$$T: V \longrightarrow W, \quad T(v) = \bar{0} \quad \forall\ v \in V$$

Identity Linear Transformation

$$T: V \longrightarrow W, \quad T(v) = v \quad \forall\ v \in V$$

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☑ Kernal & Range of a Linear Transformation

Let $T: V \longrightarrow W$ be a L.T. then,

Kernal of T is, $N(T) = \ker T = \{ x \in V : T(x) = 0 \}$

→ N(T) can't be empty
→ N(T) is always subspace of V(domain)
→ also called Null Space
→ $n(T) = \dim(N(T)) \Rightarrow$ nullity

Range Space: $R(T) = \langle T(x) \in W : x \in V \rangle$

→ R(T) subset of Co-domain
→ R(T) = Co-domain if it is onto
→ R(T) = subspace of co-domain
→ $\dim(R(T))$ = rank of $T = \rho(T)$.

☑ Rank-Nullity Theorem:

$$\text{Nullity} + \text{Rank}(T) = \dim(V)$$
$$\text{ie} \quad n(T) + \rho(T) = \dim(V)$$

Let V and W be Vector Space of equal and finite dimention, T be a linear transformation, then

i) T is one-one
ii) T is onto
iii) $\text{Rank}(T) = \dim(V)$

☑ Eigen Value and Eigen Vectors

for a square matrix 'A', λ is called its eigen value if $\exists$ a column vector 'x' such that—

$$Ax = \lambda x$$
$$Ax - \lambda x = 0$$
$$\Rightarrow \boxed{(A - \lambda I)x = 0}$$

Characteristic Equation: $|A - \lambda I| = 0$ → roots of this eqn are called eigen values

now, $(A-\lambda I)X = 0$
$\longrightarrow$ Eigen Vector

* for distinct eigen values, eigen vectors will be distinct.

☑ Properties of Eigen Values

(i) A is Singular $\longleftrightarrow$ $\lambda = 0$

(ii) A is Non-Singular ~~iff~~ all λ's are non-zero

(iii) $\sum_{i=1}^{n} \lambda_i$ = trace of Matrix

(iv) $\prod_{i=1}^{n} \lambda_i = \det(A)$

(v) A and A' have same λ's

(vi) $\lambda \longrightarrow$ Eigen value of non-singular matrix A
then, eigen value of adjoint of A, $\mathrm{adj}(A)$ is $\frac{|A|}{\lambda}$

(vii) λ's for hermition matrix are all Real

(viii) λ's for Real Sym " are all Real

(ix) λ's for Skew " " are 0 or ~~pre~~ Purely imaginary

(x) λ's for Orthogonal Matrix are mod 1 ie $|\lambda| = 1$.

☑ Properties of Eigen Vectors

$\longrightarrow$ Eigen Vectors are not unique

$\longrightarrow$ if two or more eigen values are unique it may/maynot be possible to get L.I. Eigen Vectors.

$\longrightarrow$ to a given eigen vectors, there can't be two diff eigen values.

☑ Cayley Hamilton Theorem

Every Square Matrix satisfies it's own characteristic eqn.

$A_{n\times n} \longrightarrow$ Matrix

So, $A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots + a_n I_n = 0$

$\longrightarrow$ used to find A^{-1}